

R fluids

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Abstract

A theory of collisionless fluids is developed in a unified picture, where nonrotating ($\widetilde{\Omega}_1 = \widetilde{\Omega}_2 = \widetilde{\Omega}_3 = 0$) figures with anisotropic ($\sigma_{11} = \sigma_{22} = \sigma_{33}$) random velocity component distributions and rotating ($\widetilde{\Omega}_1 \neq \widetilde{\Omega}_2 \neq \widetilde{\Omega}_3$) figures with isotropic ($\sigma_{11} \neq \sigma_{22} \neq \sigma_{33}$) random velocity component distributions, make adjoints configurations to the same system. R fluids are defined as ideal, self-gravitating fluids satisfying the virial theorem assumptions, in presence of systematic rotation around each principal axis of inertia. To this aim, mean and rms angular velocities and mean and rms tangential velocity components are expressed, by weighting on the moment of inertia and the mass, respectively. The figure rotation is defined as the mean angular velocity, weighted on the moment of inertia, with respect to a selected axis. The generalized tensor virial equations (Caimmi & Marmo 2005) are formulated for R fluids and further attention is devoted to axisymmetric configurations where, for selected coordinate axes, a variation in figure rotation has to be counterbalanced by a variation in anisotropy excess and vice versa. A microscopical analysis of systematic and random motions is performed under a few general

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hypotheses, by reversing the sign of tangential or axial velocity components of an assigned fraction of particles, leaving the distribution function and other parameters unchanged (Meza 2002). The application of the reversion process to tangential velocity components, is found to imply the conversion of random motion rotation kinetic energy into systematic motion rotation kinetic energy. The application of the reversion process to axial velocity components, is found to imply the conversion of random motion translation kinetic energy into systematic motion translation kinetic energy, and the loss related to a change of reference frame is expressed in terms of systematic (imaginary) motion rotation kinetic energy. A number of special situations are investigated with further detail. It is found that a R fluid always admits an adjoint configuration where figure rotation occurs around only one principal axis of inertia (R3 fluid), which implies that all the results related to R3 fluids (Caimmi 2007) may be extended to R fluids. Finally, a procedure is sketched for deriving the spin parameter distribution (including imaginary rotation) from a sample of observed or simulated large-scale collisionless fluids i.e. galaxies and galaxy clusters.

keywords - galaxies: clusters - galaxies: haloes - stars: stellar systems.

1 Introduction

Due to particle shocks, collisional fluids (e.g., stars, gas clouds) exhibit an isotropic stress tensor ($\sigma_{11}^2 = \sigma_{22}^2 = \sigma_{33}^2$), where σ_{pp}^2 is the rms random velocity component on the axis, x_p . The absence of particle shocks (leaving aside extreme situations, such as high-density galactic nuclei) makes a different situation in collisionless fluids (e.g., galaxies, galaxy clusters), where the stress tensor is - in general - anisotropic ($\sigma_{11}^2 \neq \sigma_{22}^2 \neq \sigma_{33}^2$). The shape of the body is determined by systematic rotation, which is quantified by a spin parameter (null for nonrotating configurations), and/or by the difference between stress tensor diagonal components, or any equivalent anisotropy indicator, related to the rotation and an equatorial principal axis of inertia, respectively (null for configurations where the random velocity component distribution is isotropic). A description of collisionless fluids based on the equivalence of systematic and random motions with respect to the shape, appears to be highly rewarding and it would provide further insight on the

properties of stellar and galaxy systems.

In an earlier attempt (Caimmi 1996) the stress tensor has been expressed as the sum of two terms, one related to a random (isotropic) velocity component distribution, and one other to anisotropic internal motions within the system. Further investigation has been devoted to the simplest situation where the system is made of two equal components, which are rotating at the same rate but in opposite sense. Then it has been recognized that the anisotropy excess may be related to real rotation, if the shape is flattened, and to imaginary rotation, if the shape is elongated, with respect to the rotation axis.

A latter approach (Caimmi & Marmo 2005) has been restricted to homeoidally striated density profiles, for which the tensor virial equations were formulated and generalized to unrelaxed configurations. The kinetic-energy tensor has been expressed as the sum of two terms, one related to systematic rotation obeying an assigned law, and one other to the remaining motions e.g., random motions, streaming motions, radial motions. Finally, an expression of the spin parameter in terms of the anisotropy excess, has shown the role of systematic and remaining motions in flattening or elongating the shape.

The above mentioned results have been improved and extended in subsequent work (Caimmi 2006, hereafter quoted as C06), where imaginary rotation has been related to negative anisotropy excess. Then sequences of configurations for which the generalized tensor virial equations hold, have been determined for homeoidally striated Jacobi ellipsoids including prolate shapes induced by imaginary rotation. The results from numerical simulations on the stability of rapidly rotating spherical configurations (Meza 2002) have been interpreted in the light of the theory. To this respect, the key argument is that the reversion (from clockwise to counterclockwise or vice versa) of tangential velocity components related to an assigned fraction of particles, preserves the potential energy, the kinetic energy, and the distribution function (Lynden-Bell 1960, 1962; Meza 2002).

The study on homeoidally striated Jacobi ellipsoids has been extended to a more general class of bodies (R3 fluids) in a recent paper (Caimmi 2007, hereafter quoted as C07), where the contribution of radial and tangential velocity components on the equatorial plane was investigated with further detail. In addition, mean and rms (weighted on the moment of inertia) angular velocity have been defined, and related to systematic and random motion tangential kinetic-energy tensor components, respectively. Also for R3 fluids,

it has been realized that the effect of (positive or negative) anisotropy excess is equivalent to additional (real or imaginary) figure rotation.

The current attempt is aimed to extend the above mentioned results to a still more general class of bodies, R fluids, defined as ideal, self-gravitating, collisionless fluids where rotation occurs around each principal axis of inertia. It will be found that R fluids always admit an adjoint configuration where figure rotation occurs around a single principal axis, that is a R3 fluid. Accordingly, all the results which hold for R3 fluids may be extended to R fluids.

The work is organized as follows. A number of basic definitions are provided in Sect. 2, including the inertia tensor, the angular-velocity tensor, and the angular-momentum tensor. The generalized tensor virial equations for R fluids are formulated in Sect. 3. The microscopical analysis of systematic and random motions, for a collisionless fluid made of N identical particles, is performed in Sect. 4, where a velocity component reversion process is defined, and a number of special situations are analysed in detail with respect to kinetic energy changes from random to systematic motions and vice versa. A procedure aimed to the derivation of the spin parameter distribution (including imaginary rotation) from an assigned sample of observed or simulated objects, is outlined in Sect. 5. Some concluding remarks are reported in Sect. 6.

2 Angular-velocity and angular-momentum tensor

In the special case of solid bodies, rotation is rigid and occurs around a single axis which, in turn, can remain fixed or change its direction. Accordingly, the angular momentum and the rotation kinetic energy read (e.g., Landau & Lifchits 1966, Chap. VI, §§ 31-33; hereafter quoted as LL66):

$$J_r = \sum_{s=1}^3 I'_{rs} \Omega_s \quad ; \quad r = 1, 2, 3 \quad ; \quad (1)$$

$$T_{\text{rot}} = \frac{1}{2} \sum_{r=1}^3 \sum_{s=1}^3 I'_{rs} \Omega_r \Omega_s \quad ; \quad (2)$$

where $\vec{J} = (J_1, J_2, J_3)$ is the angular-momentum vector, $\vec{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ the angular-velocity vector, and I' the inertia tensor:

$$I'_{rs} = \int_S \rho(x_1, x_2, x_3) \left[\delta_{rs} \sum_{r=1}^3 x_r^2 - x_r x_s \right] d^3S ; \quad (3)$$

related to the density profile, ρ , within the volume, S , being δ_{rs} the Kronecker symbol. The diagonal components of the inertia tensor, I'_{11} , I'_{22} , I'_{33} , are the moments of inertia with respect to the axes, x_1 , x_2 , x_3 , respectively.

In addition, the inertia tensor is symmetric of second rank, which implies the existence of a reference frame, $(O' X_1 X_2 X_3)$, where the inertia tensor is diagonal (LL66, Chap. VI, § 32):

$$I'_{rs} = \delta_{rs} I'_{rr} ; \quad (4)$$

the coordinate axes coincide with the principal axes of inertia, and the diagonal components define the principal moments of inertia. Accordingly, Eqs. (1) and (2) reduce to:

$$J_r = \sum_{s=1}^3 I'_{rr} \Omega_r ; \quad r = 1, 2, 3 ; \quad (5)$$

$$T_{\text{rot}} = \frac{1}{2} \left(I'_{11} \Omega_1^2 + I'_{22} \Omega_2^2 + I'_{33} \Omega_3^2 \right) ; \quad (6)$$

where, in addition (LL66, Chap. VI, § 32):

$$I'_{rr} \leq I'_{ss} + I'_{tt} ; \quad r = 1, 2, 3 ; \quad s = 2, 3, 1 ; \quad t = 3, 1, 2 ; \quad (7)$$

with regard to the body under consideration.

The inertia tensor has been defined in a different way, as (e.g., Chandrasekhar 1969, Chap. 2, § 9; Binney & Tremaine 1987, Chap. 4, § 3):

$$I_{rs} = \int_S \rho(x_1, x_2, x_3) x_r x_s d^3S ; \quad (8)$$

and the combination of Eqs. (3) and (8) yields:

$$I'_{rs} = \delta_{rs} \sum_{r=1}^3 I_{rr} - I_{rs} ; \quad (9)$$

$$I'_{rr} = I_{ss} + I_{tt} ; \quad r \neq s \neq t ; \quad (10)$$

$$I'_{rs} = -I_{rs} ; \quad r \neq s ; \quad (11)$$

or:

$$2I_{rr} = I'_{ss} + I'_{tt} - I'_{rr} \quad ; \quad r \neq s \neq t \quad ; \quad (12)$$

$$I_{rs} = -I'_{rs} \quad ; \quad r \neq s \quad ; \quad (13)$$

which translates one formulation into the other (e.g., Bett et al. 2007).

In the general case of (collisional or collisionless) fluids, rotation could occur different from solid-body, and around each principal axis of inertia. Let $(O x_1 x_2 x_3)$ be a generic reference frame and $(O' X_1 X_2 X_3)$ a reference frame where the origin coincides with the centre of inertia, and the coordinate axes coincide with the principal axes of inertia. Let the coordinate axes, X_1, X_2, X_3 , be defined as the principal axes. Let $\vec{\Omega}_1, \vec{\Omega}_2, \vec{\Omega}_3$, be the angular-velocity vectors (to be specified later) related to the principal axes of inertia. Let Ω_{rs} be the component of the vector, $\vec{\Omega}_r$, on the coordinate axis, x_s . The (3×3) tensor, Ω_{rs} , is defined as the angular-velocity tensor of the system under consideration, with respect to the reference frame, $(O x_1 x_2 x_3)$. Let $\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3$, be the angular-velocity vectors related to the coordinate axes, x_1, x_2, x_3 . The following relation holds:

$$\omega_s = \sum_{r=1}^3 \Omega_{rs} \quad ; \quad s = 1, 2, 3 \quad ; \quad (14)$$

and the angular-velocity tensor, $\omega_{rs} = \Omega_{rs}$, can formally be defined. Similarly, the (3×3) angular-momentum tensor is expressed as:

$$J'_{rs} = I'_{rs} \omega_{rs} \quad ; \quad (15)$$

$$J'_s = \sum_{r=1}^3 I'_{rs} \omega_{rs} \quad ; \quad (16)$$

where the inertia tensor is related to the reference frame, $(O x_1 x_2 x_3)$.

In the special case where the reference frame, $(O x_1 x_2 x_3)$, coincides with $(O' X_1 X_2 X_3)$, then $\Omega_{rs} = \delta_{rs} \Omega_{rr}$. Accordingly, Eqs. (14)-(16) reduce to:

$$\omega_s = \Omega_s = \Omega_{ss} = \omega_{ss} \quad ; \quad (17)$$

$$J'_{rs} = \delta_{rs} I'_{rr} \Omega_{rr} \quad ; \quad (18)$$

$$J'_s = J_s = I'_{ss} \Omega_s \quad ; \quad (19)$$

where $I'_{tt} = I_{rr} + I_{ss}$, $r \neq s \neq t$, represents the moment of inertia with respect to the principal axis of inertia, x_t . From this point on, it shall be intended

that the origin coincides with the centre of inertia, and the coordinate axes coincide with the principal axes of inertia.

The rotation kinetic-energy tensor is defined as:

$$(T_{\text{rot}})_{rs} = \frac{1}{2} I'_{rs} \Omega_r \Omega_s = \frac{1}{2} \delta_{rs} I'_{rr} \Omega_r^2 ; \quad (20)$$

where the diagonal components of the angular-velocity tensor are expressed as (C07):

$$\Omega_r = \widetilde{\Omega}_r = \frac{1}{I'_{rr}} \int_S \left| \overrightarrow{\Omega}_r(x_1, x_2, x_3, t) \right| w_r^2 \rho(x_1, x_2, x_3, t) d^3 S; \quad r \neq s \neq t; \quad (21)$$

$$\left| \overrightarrow{\Omega}_r(x_1, x_2, x_3, t) \right| = \frac{v_{\phi_r}(x_1, x_2, x_3, t)}{w_r} ; \quad (22)$$

$$w_r = (x_s^2 + x_t^2)^{1/2} ; \quad (23)$$

and $\Omega_r(x_1, x_2, x_3, t)$ is the mean value related to all the particles at the time, t , within the infinitesimal volume element, $d^3 S = dx_1 dx_2 dx_3$, centred on the point, $P(x_1, x_2, x_3)$, v_{ϕ_r} is the tangential velocity component on the $(O x_s x_t)$ principal plane, and the moment of inertia, $I'_{rr} = I_{ss} + I_{tt}$, reads:

$$I'_{rr} = \int_S w_r^2 \rho(x_1, x_2, x_3, t) d^3 S ; \quad r \neq s \neq t ; \quad (24)$$

as expected from the theorem of the mean, in connection with Eq. (21).

Similarly, the mean square diagonal components of the angular-velocity tensor are expressed as:

$$\Omega_r^2 = (\widetilde{\Omega}_r^2) = \frac{1}{I'_{rr}} \int_S \left[\overrightarrow{\Omega}_r(x_1, x_2, x_3, t) \right]^2 w_r^2 \rho(x_1, x_2, x_3, t) d^3 S; \quad r \neq s \neq t; \quad (25)$$

and the related variance reads:

$$\left(\widetilde{\sigma_{\Omega_r \Omega_r}} \right)^2 = (\widetilde{\Omega_r^2}) - (\widetilde{\Omega}_r)^2 ; \quad (26)$$

as known from statistics.

At this stage, it may be useful to extend and generalize the definition of figure rotation.

Figure rotation. *Given a R fluid, the figure rotation is defined as the mean angular velocity, weighted on the moment of inertia, with respect to a selected principal axis.*

In terms of tangential velocity components, the counterparts of Eqs. (21), (25), and (26) read:

$$\widetilde{v_{\phi_r}} = \frac{1}{M} \int_S \left| \vec{\Omega}_r(x_1, x_2, x_3, t) \right| w_r \rho(x_1, x_2, x_3, t) d^3 S \quad ; \quad r \neq s \neq t \quad ; \quad (27)$$

$$(\widetilde{v_{\phi_r}^2}) = \frac{1}{M} \int_S \left[\vec{\Omega}_r(x_1, x_2, x_3, t) \right]^2 w_r^2 \rho(x_1, x_2, x_3, t) d^3 S; \quad r \neq s \neq t; \quad (28)$$

$$\left(\widetilde{\sigma_{v_{\phi_r} v_{\phi_r}}} \right)^2 = (\widetilde{v_{\phi_r}^2}) - (\widetilde{v_{\phi_r}})^2 \quad ; \quad (29)$$

and the combination of Eqs. (25) and (28); (21) and (27); (26) and (29); yields:

$$M(\widetilde{v_{\phi_r}^2}) = I'_{rr}(\widetilde{\Omega_r^2}) \quad ; \quad (30a)$$

$$M(\widetilde{v_{\phi_r}})^2 = I'_{rr}(\widetilde{\Omega_r})^2 \quad ; \quad (30b)$$

$$M\left(\widetilde{\sigma_{\phi_r \phi_r}}\right)^2 = I'_{rr}\left(\widetilde{\sigma_{\Omega_r \Omega_r}}\right)^2 \quad ; \quad (30c)$$

which relate tangential velocity components on the $(O x_s x_t)$ principal plane, to angular velocity components on the x_r principal axis.

3 The generalized tensor virial equations for R fluids

Let R fluids be defined as (collisional or collisionless) ideal self-gravitating fluids where figure rotation occurs around all the three principal axes of inertia. Let $(O x_1 x_2 x_3)$ be a reference frame where the origin coincides with the centre of inertia, and the coordinate axes coincide with the principal axes of inertia. Then the mean radial velocity components must necessarily equal zero:

$$\overline{v_{w_r}} = 0 \quad ; \quad \overline{(v_{w_r}^2)} = (\sigma_{w_r w_r})^2 \quad ; \quad (31)$$

where v_{w_r} is the radial velocity component on the $(O x_s x_t)$ principal plane, perpendicular to the x_r principal axis. Let positive and negative radial velocity components be defined as directed outwards and inwards, respectively. The same holds for the mean tangential velocity components:

$$\overline{v_{\phi_r}} = 0 \quad ; \quad \overline{(v_{\phi_r}^2)} = (\sigma_{\phi_r \phi_r})^2 \quad ; \quad (32)$$

even in presence of systematic rotation. Let positive and negative tangential velocity components be defined as rotating counterclockwise and clockwise, respectively.

The kinetic-energy tensor may be expressed as the sum of two contributions: one, related to systematic motions, and one other, related to random motions (C07). The result is:

$$T_{k_s k_t} = (T_{\text{sys}})_{k_s k_t} + (T_{\text{rdm}})_{k_s k_t} \quad ; \quad k = w, \phi \quad ; \quad (33)$$

where the terms on the right-hand side, using Eqs. (30) and (31), can be expressed as:

$$(T_{\text{sys}})_{w_s w_t} = 0 \quad ; \quad (34a)$$

$$(T_{\text{rdm}})_{w_s w_t} = \frac{1}{2} \delta_{st} M (\sigma_{w_s w_s})^2 \quad ; \quad (34b)$$

$$(T_{\text{sys}})_{\phi_s \phi_t} = \frac{1}{2} \delta_{st} I'_{ss} (\widetilde{\Omega_s})^2 \quad ; \quad (35a)$$

$$(T_{\text{rdm}})_{\phi_s \phi_t} = \frac{1}{2} \delta_{st} I'_{ss} (\sigma_{\widetilde{\Omega_s \Omega_s}})^2 \quad ; \quad (35b)$$

keeping in mind that nondiagonal components are null in the case under discussion, only diagonal components shall be considered from this point on. The combination of Eqs. (26) and (35) yields:

$$T_{\phi_r \phi_r} = \frac{1}{2} I'_{rr} (\widetilde{\Omega_r^2}) \quad ; \quad (36)$$

which depends on the density profile via the moment of inertia, I'_{rr} , and the tangential velocity component distribution via the mean square angular velocity, $(\widetilde{\Omega_r^2})$, regardless from the fraction of systematic and random motions.

In terms of the contributions related to the axial components of the kinetic-energy tensor, T_{ss} and T_{tt} , Eqs. (33), (35), and (36) read:

$$(T_{\phi_r \phi_r})_{\ell\ell} = \frac{1}{2} I_{\ell\ell} (\widetilde{\Omega_r^2}) \quad ; \quad \ell = s, t \quad ; \quad (37a)$$

$$[(T_{\text{sys}})_{\phi_r \phi_r}]_{\ell\ell} = \frac{1}{2} I_{\ell\ell} (\widetilde{\Omega_r})^2 \quad ; \quad \ell = s, t \quad ; \quad (37b)$$

$$[(T_{\text{rdm}})_{\phi_r \phi_r}]_{\ell\ell} = \frac{1}{2} I_{\ell\ell} [(\widetilde{\Omega_r^2}) - (\widetilde{\Omega_r})^2] \quad ; \quad \ell = s, t \quad ; \quad (37c)$$

where Eq. (10) has been used.

The invariance of a vector with respect to a change of the reference frame, implies the validity of the relations (C07):

$$\overline{(v_{w_r}^2)} + \overline{(v_{\phi_r}^2)} = \overline{(v_s^2)} + \overline{(v_t^2)} ; \quad (38)$$

$$(\overline{v_{w_r}})^2 + (\overline{v_{\phi_r}})^2 = (\overline{v_s})^2 + (\overline{v_t})^2 ; \quad (39)$$

$$(\sigma_{w_r w_r})^2 + (\sigma_{\phi_r \phi_r})^2 = (\sigma_{ss})^2 + (\sigma_{tt})^2 ; \quad (40)$$

where the velocity components on the x_s and x_t principal axes are labelled by the indices, s and t , respectively.

The combination of Eqs. (26), (29), and (38)-(40) yields:

$$(\sigma_{w_r w_r})^2 = (\sigma_{ss})^2 + (\sigma_{tt})^2 - \frac{I'_{rr}}{M} [(\widetilde{\Omega_r^2}) - (\widetilde{\Omega_r})^2] ; \quad (41)$$

which makes Eqs. (33) and (34) translate into:

$$(T_{w_r w_r})_{\ell\ell} = [(T_{\text{rdm}})_{w_r w_r}]_{\ell\ell} = \frac{1}{2} M \sigma_{\ell\ell}^2 - \frac{1}{2} I_{\ell\ell} [(\widetilde{\Omega_r^2}) - (\widetilde{\Omega_r})^2] ; \quad \ell = s, t ; \quad (42a)$$

$$[(T_{\text{sys}})_{w_r w_r}]_{\ell\ell} = 0 ; \quad \ell = s, t ; \quad (42b)$$

in terms of the contributions related to the axial components of the kinetic-energy tensor, T_{ss} and T_{tt} .

The generalized tensor virial equations of the second order can be formulated, extending the procedure used for R3 fluids (C07). The result is:

$$I_{rr} [(\widetilde{\Omega_s})^2 + (\widetilde{\Omega_t})^2] + M \zeta_{rr} \sigma^2 + (E_{\text{pot}})_{rr} = 0 ; \quad (43)$$

$$\sigma^2 = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 ; \quad (44)$$

$$\zeta_{pp} = \frac{(\widetilde{T}_{\text{rdm}})_{pp}}{T_{\text{rdm}}} = \frac{\sigma_{pp}^2}{\sigma^2} ; \quad p = 1, 2, 3 ; \quad (45a)$$

$$\zeta_{11} + \zeta_{22} + \zeta_{33} = \frac{\widetilde{T}_{\text{rdm}}}{T_{\text{rdm}}} = \frac{\widetilde{\sigma}^2}{\sigma^2} = \zeta ; \quad (45b)$$

where mean angular velocity components on the x_r principal axis are due to systematic rotation around x_s and x_t axes, $(E_{\text{pot}})_{rr}$ is the self potential-energy tensor, ζ_{rr} may be conceived as generalized anisotropy parameters

(Caimmi & Marmo 2005; C06; C07), and \tilde{T}_{rdm} is the effective random kinetic energy i.e. the right amount needed for an instantaneous configuration to satisfy the usual tensor virial equations of the second order, defined by the effective anisotropy parameters (C07):

$$\tilde{\zeta}_{pp} = \frac{(\tilde{T}_{\text{rdm}})_{pp}}{\tilde{T}_{\text{rdm}}} = \frac{\zeta_{pp}}{\zeta} \quad ; \quad p = 1, 2, 3 \quad ; \quad (46a)$$

$$\tilde{\zeta}_{11} + \tilde{\zeta}_{22} + \tilde{\zeta}_{33} = 1 \quad ; \quad (46b)$$

and the condition, $\zeta = 1$, or $\zeta_{pp} = \tilde{\zeta}_{pp}$, $p = 1, 2, 3$, makes Eqs. (43) reduce to their standard counterparts. To get further insight, a microscopical analysis is needed.

In the special case of axisymmetric configurations, $I_{11} = I_{22}$, $(E_{\text{pot}})_{11} = (E_{\text{pot}})_{22}$, and the combination of the related tensor virial equations, expressed by Eq. (43), yields:

$$I_{pp} [(\tilde{\Omega}_q)^2 - (\tilde{\Omega}_p)^2] = M\sigma^2(\zeta_{qq} - \zeta_{pp}) \quad ; \quad p = 1, 2 \quad ; \quad q = 2, 1 \quad ; \quad (47)$$

where a figure rotation excess, $[(\tilde{\Omega}_q)^2 - (\tilde{\Omega}_p)^2]$, is counterbalanced by an anisotropy excess, $(\zeta_{qq} - \zeta_{pp})$; in particular, a null figure rotation excess implies a null anisotropy excess and vice versa. Accordingly, a flattening on the $(O x_p x_r)$ principal plane, induced by the figure rotation excess, has to be counterbalanced by an elongation on the x_q principal axis, induced by the anisotropy excess, to yield an axisymmetric configuration with respect to the x_r principal axis.

4 Microscopical analysis of systematic and random motions

Given a collisionless R fluid, let N be the total number of particles and m the mean particle mass in absence of mass segregation i.e. local and global mean particle mass coincide. For simplicity, the equivalent description (C07) involving N identical particles of mass, m , shall be considered. With regard to the $(O x_s x_t)$ principal plane, let v_{ϕ_r} be the tangential velocity component on the above mentioned plane. It is worth noting (Meza 2002) that the distribution function is independent of the sign of v_{ϕ_r} , and the

whole set of possible configurations are characterized by an equal amount of both kinetic and potential energy. Numerical simulations show that spherical systems, even if rapidly rotating, are dynamically stable after reversion of the tangential velocity component in an assigned fraction of particles (Meza 2002).

For sake of simplicity, let the initial configuration be nonrotating ($\overline{v_{\phi_r}} = 0$) and with isotropic random velocity component distribution ($\zeta_{11} = \zeta_{22} = \zeta_{33}$). In the case under discussion of identical particles, $m^{(i)} = m$, $1 \leq i \leq N$, the centre of inertia velocity components, v_{Cr} , equal the related arithmetic means:

$$v_{Cr} = \frac{\sum_{i=1}^N m^{(i)} v_r^{(i)}}{\sum_{i=1}^N m^{(i)}} = \frac{m \sum_{i=1}^N v_r^{(i)}}{Nm} = \overline{v_r} \quad ; \quad (48)$$

and the moments of inertia, I'_{rr} , reduce to:

$$I'_{rr} = \sum_{i=1}^N m^{(i)} [w_r^{(i)}]^2 = m \sum_{i=1}^N [w_r^{(i)}]^2 = MR_{Gr}^2 \quad ; \quad (49)$$

where $R_{Gr} = [\overline{w_r^2}]^{1/2}$ is the curl radius with respect to the x_r axis.

The weighted mean, mean square, and rms tangential velocity components, expressed by Eqs. (27)-(29), read:

$$\widetilde{v_{\phi_r}} = \frac{1}{M} \sum_{i=1}^N m^{(i)} v_{\phi_r}^{(i)} = \frac{m}{M} \sum_{i=1}^N v_{\phi_r}^{(i)} = \overline{v_{\phi_r}} \quad ; \quad (50)$$

$$(\widetilde{v_{\phi_r}^2}) = \frac{1}{M} \sum_{i=1}^N m^{(i)} [v_{\phi_r}^{(i)}]^2 = \frac{m}{M} \sum_{i=1}^N [v_{\phi_r}^{(i)}]^2 = \overline{(v_{\phi_r}^2)} \quad ; \quad (51)$$

$$(\sigma_{\phi_r \phi_r})^2 = \overline{(v_{\phi_r}^2)} - (\overline{v_{\phi_r}})^2 = (\sigma_{\phi_r \phi_r})^2 \quad ; \quad (52)$$

and Eqs. (30) reduce to:

$$\overline{(v_{\phi_r}^2)} = R_{Gr}^2 (\widetilde{\Omega_r^2}) \quad ; \quad (53a)$$

$$(\overline{v_{\phi_r}})^2 = R_{Gr}^2 (\widetilde{\Omega_r})^2 \quad ; \quad (53b)$$

$$(\sigma_{\phi_r \phi_r})^2 = R_{Gr}^2 (\sigma_{\widetilde{\Omega_r \Omega_r}})^2 \quad ; \quad (53c)$$

which relate weighted angular velocities around the x_r axis to mean tangential velocity components on the $(O x_s x_t)$ principal plane.

At this stage, let the tangential velocity component of a fraction, n/N , of particles, be reversed in equal sense (from clockwise to counterclockwise or vice versa), according to the following assumptions.

- (i) Both the number, n , of particles where the tangential velocity component has been reversed, and the number, $N - n$, of particles which remain unchanged, are sufficiently large, $1 \ll n \ll N$, $0 \leq n \leq \text{Int}(N/2)$.
- (ii) The fraction, n_k/N_k , of particles where the tangential velocity component has been reversed, within a generic volume element, S_k , is independent of the volume element, $n_k/N_k = n/N$.
- (iii) The system is made of identical particles, $m^{(i)} = m$, $M = mN$.
- (iv) After tangential velocity components have been reversed in n_k particles within a generic volume element, S_k , on a total of N_k , a second set of n_k particles (among the remaining $N_k - n_k$) exists, where the tangential velocity component of any particle equals its counterpart belonging to the first set.

In the following, the above process shall be quoted as “the reversion process”.

Obviously, mean square tangential velocity components, $\overline{(v_{\phi_r}^2)}$, are left unchanged by the reversion process. On the contrary, mean tangential velocity components after the reversion process read:

$$\overline{v_{\phi_r}} = \frac{1}{N} \sum_{i=1}^N v_{\phi_r}^{(i)} = \frac{1}{N} \left[\sum_{i=1}^{2n} v_{\phi_r}^{(i)} + \sum_{i=2n+1}^N v_{\phi_r}^{(i)} \right] ; \quad (54)$$

where the first sum within brackets relates to particles where the reversion process has occurred and their counterparts with equal tangential velocity components, while the second sum comprises the remaining particles and necessarily equals the mean tangential velocity component before the occurrence of the reversion process, which is null in the case under discussion. Accordingly, Eq. (54) reduces to:

$$\overline{v_{\phi_r}} = \frac{2n}{N} (\overline{v_{\phi_r}})_n ; \quad (55a)$$

$$(\overline{v_{\phi_r}})_n = \frac{1}{2n} \sum_{i=1}^{2n} v_{\phi_r}^{(i)} = \frac{1}{n} \sum_{i=1}^n v_{\phi_r}^{(i)} ; \quad (55b)$$

keeping in mind that the first sum is performed on couples of particles with equal tangential velocity components.

The validity of Eqs. (54) and (55) still maintains if tangential velocity components, v_{ϕ_r} , are replaced by axial velocity components, v_r . The combination of Eqs. (48) and (55a) yields:

$$v_{Cr} = \overline{v_r} = \frac{2n}{N}(\overline{v_r})_n ; \quad (56)$$

which is the velocity component of the centre of inertia with respect to the x_r principal axis, after the reversion process.

The total kinetic energy is left unchanged by the reversion process but, on the other hand, a fraction of random motion kinetic energy is turned into systematic motion kinetic energy. In the following, the reversion process shall be discussed with further details for a number of different situations.

4.1 Tangential velocity component reversion

Performing the reversion process on a given fraction of particles, n/N , with respect to tangential velocity components, implies the conversion of random (rotation) motion kinetic energy into systematic (rotation) motion kinetic energy, as:

$$\Delta(T_{\text{rdm}})_{\phi_r\phi_r} = -\Delta(T_{\text{sys}})_{\phi_r\phi_r} = -\frac{1}{2}M(\overline{v_{\phi_r}})^2 = -\frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 ; \quad (57)$$

where the remaining parameters are left unchanged.

The occurrence of the reversion process implies the following energy changes:

$$T_{\text{rdm}} \rightarrow T_{\text{rdm}} - \frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 ; \quad (58)$$

$$(T_{\text{rdm}})_{\phi_r\phi_r} \rightarrow (T_{\text{rdm}})_{\phi_r\phi_r} - \frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 ; \quad (59)$$

$$(T_{\text{rdm}})_{\ell\ell} \rightarrow (T_{\text{rdm}})_{\ell\ell} - \frac{1}{2}\frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 ; \quad \ell = s, t ; \quad (60)$$

$$T_{\text{sys}} \rightarrow 0 + \frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 ; \quad (61)$$

$$(T_{\text{sys}})_{\phi_r\phi_r} \rightarrow 0 + \frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 ; \quad (62)$$

$$(T_{\text{sys}})_{\ell\ell} \rightarrow 0 + \frac{1}{2} \frac{2n}{N} nm(\overline{v_{\phi_r}})_n^2 ; \quad \ell = s, t ; \quad (63)$$

while the contributions from random radial motions along the equatorial plane, $(T_{\text{rdm}})_{w_r w_r}$, and the rotation axis, $(T_{\text{rdm}})_{rr}$, remain unchanged.

With the system being relaxed, $\zeta = 1$, in the case under discussion, the generalized and effective anisotropy parameters, ζ_{pp} and $\tilde{\zeta}_{pp}$, coincide with their counterparts related to the usual tensor virial equations, and Eqs. (45a) and (46a) take the explicit form (C06):

$$\zeta_{\ell\ell} = \frac{(1/3)T_{\text{rdm}} - (2n/N)(n/2)m[(\overline{v_{\phi_r}})_n]^2}{T_{\text{rdm}} - (2n/N)nm[(\overline{v_{\phi_r}})_n]^2} ; \quad \ell = s, t ; \quad (64a)$$

$$\zeta_{rr} = \frac{(1/3)T_{\text{rdm}}}{T_{\text{rdm}} - (2n/N)nm[(\overline{v_{\phi_r}})_n]^2} ; \quad (64b)$$

where the special case, $n = 0$, relates to the initial configuration, characterized by isotropic random velocity component distributions ($\zeta_{pp} = 1/3$) and no figure rotation.

In the extreme case where the reversion process is completed, $n = N/2$, the changes expressed by Eqs. (58)-(63) take the form:

$$T_{\text{rdm}} \rightarrow T_{\text{rdm}} - \frac{N}{2} m[(\overline{v_{\phi_r}})_{N/2}]^2 ; \quad (65)$$

$$(T_{\text{rdm}})_{\phi_r \phi_r} \rightarrow (T_{\text{rdm}})_{\phi_r \phi_r} - \frac{N}{2} m[(\overline{v_{\phi_r}})_{N/2}]^2 ; \quad (66)$$

$$(T_{\text{rdm}})_{\ell\ell} \rightarrow (T_{\text{rdm}})_{\ell\ell} - \frac{N}{4} m[(\overline{v_{\phi_r}})_{N/2}]^2 ; \quad \ell = s, t ; \quad (67)$$

$$T_{\text{sys}} \rightarrow 0 + \frac{N}{2} m[(\overline{v_{\phi_r}})_{N/2}]^2 ; \quad (68)$$

$$(T_{\text{sys}})_{\phi_r \phi_r} \rightarrow 0 + \frac{N}{2} m[(\overline{v_{\phi_r}})_{N/2}]^2 ; \quad (69)$$

$$(T_{\text{sys}})_{\ell\ell} \rightarrow 0 + \frac{N}{4} m[(\overline{v_{\phi_r}})_{N/2}]^2 ; \quad \ell = s, t ; \quad (70)$$

similarly, Eqs. (64) take the form:

$$\zeta_{\ell\ell} = \frac{(1/3)T_{\text{rdm}} - (N/4)m[(\overline{v_{\phi_r}})_{N/2}]^2}{T_{\text{rdm}} - (N/2)m[(\overline{v_{\phi_r}})_{N/2}]^2} ; \quad \ell = s, t ; \quad (71a)$$

$$\zeta_{rr} = \frac{(1/3)T_{\text{rdm}}}{T_{\text{rdm}} - (N/2)m[(\overline{v_{\phi_r}})_{N/2}]^2} ; \quad (71b)$$

in any case, the anisotropy excess, $\zeta_{\ell\ell} - \zeta_{rr} < 0$, is counterbalanced by figure rotation.

4.2 Axial velocity component reversion

Performing the reversion process on a given fraction of particles, n/N , with respect to axial velocity components, implies the conversion of random (translation) motion kinetic energy into systematic (translation) motion kinetic energy, as:

$$\Delta(T_{\text{rdm}})_{rr} = -\Delta(T_{\text{sys}})_{rr} = -\frac{1}{2}M(\overline{v_r})^2 = -\frac{2n}{N}nm[(\overline{v_r})_n]^2 ; \quad (72)$$

where the remaining parameters are left unchanged.

The reversion process implies the following energy changes:

$$T_{\text{rdm}} \rightarrow T_{\text{rdm}} - \frac{2n}{N}nm[(\overline{v_r})_n]^2 ; \quad (73)$$

$$(T_{\text{rdm}})_{rr} \rightarrow (T_{\text{rdm}})_{rr} - \frac{2n}{N}nm[(\overline{v_r})_n]^2 ; \quad (74)$$

$$T_{\text{sys}} \rightarrow 0 + \frac{2n}{N}nm[(\overline{v_r})_n]^2 ; \quad (75)$$

$$(T_{\text{sys}})_{rr} \rightarrow 0 + \frac{2n}{N}nm[(\overline{v_r})_n]^2 ; \quad (76)$$

while the contributions from random motions along the x_s and x_t principal axis, $(T_{\text{rdm}})_{ss}$ and $(T_{\text{rdm}})_{tt}$, remain unchanged.

With the system being relaxed, $\zeta = 1$, in the case under discussion, the generalized and effective anisotropy parameters, ζ_{pp} and $\tilde{\zeta}_{pp}$, coincide with their counterparts related to the usual tensor virial equations, and Eqs. (45a) and (46a) take the explicit form:

$$\zeta_{\ell\ell} = \frac{(1/3)T_{\text{rdm}}}{T_{\text{rdm}} - (2n/N)nm[(\overline{v_r})_n]^2} ; \quad \ell = s, t ; \quad (77a)$$

$$\zeta_{rr} = \frac{(1/3)T_{\text{rdm}} - (2n/N)nm[(\overline{v_r})_n]^2}{T_{\text{rdm}} - (2n/N)nm[(\overline{v_r})_n]^2} ; \quad (77b)$$

where the special case, $n = 0$, relates to the initial configuration, characterized by isotropic random velocity component distributions ($\zeta_{pp} = 1/3$) and no figure rotation.

In the extreme case where the reversion process is completed, $n = N/2$, the changes expressed by Eqs. (73)-(76) take the form:

$$T_{\text{rdm}} \rightarrow T_{\text{rdm}} - \frac{N}{2}m[(\overline{v_r})_{N/2}]^2 ; \quad (78)$$

$$(T_{\text{rdm}})_{rr} \rightarrow (T_{\text{rdm}})_{rr} - \frac{N}{2}m[(\overline{v_r})_{N/2}]^2 ; \quad (79)$$

$$T_{\text{sys}} \rightarrow 0 + \frac{N}{2}m[(\overline{v_r})_{N/2}]^2 ; \quad (80)$$

$$(T_{\text{sys}})_{rr} \rightarrow 0 + \frac{N}{2}m[(\overline{v_r})_{N/2}]^2 ; \quad (81)$$

similarly, Eqs. (77) take the form:

$$\zeta_{\ell\ell} = \frac{(1/3)T_{\text{rdm}}}{T_{\text{rdm}} - (N/2)m[(\overline{v_r})_{N/2}]^2} ; \quad \ell = s, t ; \quad (82a)$$

$$\zeta_{rr} = \frac{(1/3)T_{\text{rdm}} - (N/2)m[(\overline{v_r})_{N/2}]^2}{T_{\text{rdm}} - (N/2)m[(\overline{v_r})_{N/2}]^2} ; \quad (82b)$$

in any case, the anisotropy excess, $\zeta_{\ell\ell} - \zeta_{rr} > 0$, is counterbalanced by (imaginary) figure rotation.

4.3 Change of reference frame and imaginary rotation

Performing the reversion process on a given fraction of particles, n/N , implies the conversion of random motion (translation) kinetic energy into systematic motion (translation) kinetic energy, with respect to the x_r principal axis, according to Eqs. (72)-(76). The related kinetic-energy tensor component of the centre of inertia, by use of Eqs. (56) and (72) reads:

$$(T_C)_{rr} = \frac{1}{2}M(\overline{v_r})^2 = \frac{2n}{N}nm[(\overline{v_r})_n]^2 = -\Delta(T_{\text{rdm}})_{rr} ; \quad (83)$$

which, in the case under discussion, coincides with the kinetic energy of the centre of inertia, T_C , being $(T_C)_{ss} = (T_C)_{tt} = 0$. In the centre of inertia reference frame after the reversion process, the random motion kinetic-energy tensor component related to the x_r axis, $(T'_{\text{rdm}})_{rr}$, by use of Eqs. (56) and (83), takes the expression:

$$(T'_{\text{rdm}})_{rr} = \frac{1}{2}m \sum_{i=1}^N [v_r^{(i)} - v_{Cr}]^2 = (T_{\text{rdm}})_{rr} - \frac{1}{2}M(\overline{v_r})^2 = (T_{\text{rdm}})_{rr} - (T_C)_{rr} ; \quad (84)$$

where the kinetic energy, $T_C = (T_C)_{rr}$, is hidden by the change of reference frame (e.g., LL66, Chap. II, § 8).

Let the i -th particle be at the distance, $w_r^{(i)} = \{[x_s^{(i)}]^2 + [x_t^{(i)}]^2\}^{1/2}$, from the x_r principal axis, with velocity component, $v_r^{(i)}$. The imaginary angular velocity (Caimmi 1996; C06; C07), $i\Omega_r^{(i)}$, can be defined in such a way the translational kinetic energy along the x_r axis is counterbalanced by the imaginary rotational kinetic energy around the x_r axis, as:

$$\frac{1}{2}m[v_r^{(i)}]^2 + \frac{1}{2}m[w_r^{(i)}]^2 [i\Omega_r^{(i)}]^2 = 0 \quad ; \quad (85)$$

$$\Omega_r^{(i)} = \frac{v_r^{(i)}}{w_r^{(i)}} \quad ; \quad (86)$$

where the index, i , is related to the i -th particle, and the factor, i , is the imaginary unit. In this view, the velocity components on the x_r principal axis may be translated into imaginary tangential velocity components on the $(O x_s x_t)$ principal plane, as:

$$(iv_{\phi_r})^2 = (iw_r\Omega_r)^2 = -v_r^2 \quad ; \quad (87)$$

according to Eqs. (85) and (86).

Let imaginary rotation around the x_r axis be imparted to the particles where the reversion process has been occurred, and their counterparts with equal imaginary tangential velocity components, as prescribed by Eq. (87) particularized to the mean axial velocity component, $\overline{v_r}$, expressed by Eq. (56). The related increment in imaginary kinetic energy reads:

$$\Delta(T_{\text{sys}})_{\phi_r\phi_r} = \frac{1}{2}M(\overline{iv_{\phi_r}})^2 = \frac{2n}{N}nm[(\overline{iv_{\phi_r}})_n]^2 \quad ; \quad (88)$$

and the combination of Eqs. (72), (83), (87), and (88) yields:

$$\Delta(T_{\text{sys}})_{\phi_r\phi_r} = -\Delta(T_{\text{sys}})_{rr} = -(T_C)_{rr} = \Delta(T_{\text{rdm}})_{rr} \quad ; \quad (89)$$

the above results may be reduced to a single statement.

Theorem 1. *Given a R fluid with isotropic random velocity distribution and no figure rotation, let the axial velocity component reversion process be performed on a given fraction of particles, n/N , with respect to the x_r*

principal axis. Then turning to the centre of inertia reference frame with a kinetic energy loss, $\Delta(T_{\text{sys}})_{rr} = (T_C)_{rr}$, is equivalent to put the initial configuration into imaginary rotation around the x_r principal axis, with square mean tangential velocity component, $(\overline{iv_{\phi_r}})^2 = -(\overline{v_r})^2$, with a kinetic energy gain, $\Delta(T_{\text{sys}})_{\phi_r\phi_r} = -(T_C)_{rr}$.

Accordingly, Eqs. (73)-(76) are replaced by the following:

$$T_{\text{sys}} \rightarrow 0 - \frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 \quad ; \quad (90)$$

$$(T_{\text{sys}})_{\phi_r\phi_r} \rightarrow 0 - \frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 \quad ; \quad (91)$$

$$(T_{\text{sys}})_{\ell\ell} \rightarrow 0 - \frac{1}{2} \frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 \quad ; \quad (92)$$

while the contributions from random motions remain unchanged, and the random velocity distribution remains isotropic ($\zeta_{11} = \zeta_{22} = \zeta_{33} = 1/3$).

In the extreme case where the reversion process is complete, $n = N/2$, and the maximum amount of available imaginary rotation has been attained, the changes expressed by Eq. (90)-(92) take the form:

$$T_{\text{sys}} \rightarrow 0 - \frac{N}{2}m[(\overline{v_{\phi_r}})_{N/2}]^2 \quad ; \quad (93)$$

$$(T_{\text{sys}})_{\phi_r\phi_r} \rightarrow 0 - \frac{N}{2}m[(\overline{v_{\phi_r}})_{N/2}]^2 \quad ; \quad (94)$$

$$(T_{\text{sys}})_{\ell\ell} \rightarrow 0 - \frac{N}{4}m[(\overline{v_{\phi_r}})_{N/2}]^2 \quad ; \quad (95)$$

the above results may be reduced to a single statement.

Theorem 2. *Given a R fluid with isotropic random velocity distribution and no figure rotation, let the axial velocity component reversion process be performed on a given fraction of particles, n/N , with respect to the x_r principal axis, and the reference frame changed into the centre of inertia reference frame. Then the resulting configuration with anisotropy excess, $\zeta_{\ell\ell} - \zeta_{rr} > 0$, Eqs. (77), is equivalent to the initial configuration with null anisotropy excess, $\zeta_{\ell\ell} - \zeta_{rr} = 0$, and imaginary rotation around the x_r principal axis, with square mean tangential velocity component, $(\overline{iv_{\phi_r}})^2 = -(\overline{v_r})^2$.*

4.4 Anisotropy excess and imaginary rotation

Given a nonrotating ($\widetilde{\Omega}_r = 0$) isotropic ($\zeta_{11} = \zeta_{22} = \zeta_{33} = 1/3$) configuration, let the tangential velocity component on the ($\mathbf{O} x_s x_t$) principal plane be reversed in such a way Eqs. (65)-(71) hold. Let an equal amount of real and imaginary tangential velocity component on the ($\mathbf{O} x_s x_t$) principal plane, be imparted to each particle for a total contribution equal to $(2/3)f_r T_{\text{rdm}}$ and $-(2/3)f_r T_{\text{rdm}}$, respectively, where f_r is a positive real number, which leaves the total energy unchanged. Let the reversion process be repeated for real tangential velocity components, to attain a null (real) figure rotation. The related changes, with respect to the initial configuration, are:

$$T_{\text{rdm}} \rightarrow \left(1 + \frac{2}{3}f_r\right) T_{\text{rdm}} ; \quad (96)$$

$$(T_{\text{rdm}})_{\phi_r \phi_r} \rightarrow \left(\frac{2}{3} + \frac{2}{3}f_r\right) T_{\text{rdm}} ; \quad (97)$$

$$(T_{\text{rdm}})_{\ell\ell} \rightarrow \frac{1}{2} \left(\frac{2}{3} + \frac{2}{3}f_r\right) T_{\text{rdm}} ; \quad (98)$$

$$T_{\text{sys}} \rightarrow 0 - \frac{2}{3}f_r T_{\text{rdm}} ; \quad (99)$$

$$(T_{\text{sys}})_{\phi_r \phi_r} \rightarrow 0 - \frac{2}{3}f_r T_{\text{rdm}} ; \quad (100)$$

$$(T_{\text{sys}})_{\ell\ell} \rightarrow 0 - \frac{1}{2} \frac{2}{3}f_r T_{\text{rdm}} ; \quad (101)$$

and the related anisotropy parameters read:

$$\zeta_{\ell\ell} = \frac{[(1/3) + (1/3)f_r]T_{\text{rdm}}}{[1 + (2/3)f_r]T_{\text{rdm}}} = \frac{1 + f_r}{3 + 2f_r} ; \quad \ell = s, t ; \quad (102a)$$

$$\zeta_{rr} = \frac{(1/3)T_{\text{rdm}}}{[1 + (2/3)f_r]T_{\text{rdm}}} = \frac{1}{3 + 2f_r} ; \quad (102b)$$

where the anisotropy excess, $\zeta_{\ell\ell} - \zeta_{rr} = f_r/(3 + 2f_r) > 0$, is counterbalanced by imaginary rotation, according to an initial configuration with isotropic velocity component distribution and no figure rotation. As application of the above results, two significative examples shall be taken into consideration.

First example. *Nonrotating systems flattened by anisotropic velocity component distribution* ($\sigma_{11} = \sigma_{22} > \sigma_{33}$).

Let tangential velocity components on the $(\mathbf{O} x_1 x_2)$ principal plane be reversed in a convenient fraction of particles, n/N , to yield a convenient amount of figure rotation together with isotropic velocity component distribution ($\sigma'_{11} = \sigma'_{22} = \sigma_{33}$), as sketched in Fig. 1.

The combination of Eqs. (52), (53), and (55a) yields:

$$(\widetilde{\Omega}_r)^2 = \frac{1}{R_{Gr}^2} \frac{4n^2}{N^2} [(\overline{v_{\phi_r}})_n]^2 ; \quad (103)$$

$$(\widetilde{\sigma_{\Omega_r \Omega_r}})^2 = \frac{1}{R_{Gr}^2} (\sigma_{\phi_r \phi_r})^2 = \frac{1}{R_{Gr}^2} \left\{ \overline{(v_{\phi_r}^2)} - \frac{4n^2}{N^2} [(\overline{v_{\phi_r}})_n]^2 \right\} ; \quad (104)$$

while the mean square velocity components are left unchanged by the occurrence of the reversion process. The substitution of Eqs. (103) and (104) into (37b) and (37c) shows the dependence of systematic and random motion tangential kinetic-energy tensor components on the reversion process.

In the case under discussion ($r = 3$), the random velocity component distribution has to be isotropic after the reversion process, which makes Eqs. (60) and (63) reduce to:

$$(T_{\text{rdm}})_{\ell\ell} = \frac{M}{2} [\sigma_{\ell\ell}^2 - (\sigma_{\ell\ell}^2 - \sigma_{33}^2)] = \frac{M}{2} \sigma_{33}^2 ; \quad \ell = 1, 2 ; \quad (105)$$

$$(T_{\text{sys}})_{\ell\ell} = \frac{M}{2} (\sigma_{\ell\ell}^2 - \sigma_{33}^2) ; \quad \ell = 1, 2 ; \quad (106)$$

$$\frac{2n}{N} nm [\overline{(v_{\phi_r})_n}]^2 = \frac{1}{2} \frac{4n^2}{N^2} M [(\overline{v_{\phi_r}})_n]^2 = \frac{M}{2} (\overline{v_{\phi_r}})^2 = \frac{M}{2} (\sigma_{\ell\ell}^2 - \sigma_{33}^2) ; \quad (107)$$

which defines the configuration after the reversion process.

Second example. *Systems elongated by anisotropic velocity component distribution ($\sigma_{11} = \sigma_{22} < \sigma_{33}$) with no figure rotation.*

Let a convenient amount of real and imaginary figure rotation, $\overline{v_{\phi_3}}$ and $\overline{iv_{\phi_3}}$, with respect to the x_3 principal axis, be imparted to the system. Then the kinetic energy remains unchanged. Concerning real rotation, let the reversion process be performed on one half particles, leaving imaginary figure rotation together with isotropic velocity component distribution ($\sigma'_{11} = \sigma'_{22} = \sigma_{33}$), as sketched in Fig. 2. Accordingly, Eqs. (103) and (104) hold for imaginary tangential velocity components on the $(\mathbf{O} x_1 x_2)$ principal plane and imaginary figure rotation around the x_3 principal axis.

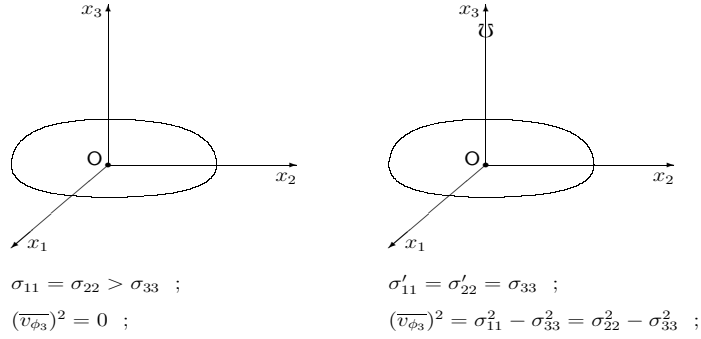


Figure 1: After reversion of tangential velocity components in a convenient fraction of particles, with respect to the (Ox_1x_2) principal plane, a configuration flattened by anisotropic velocity component distribution ($\sigma_{11} = \sigma_{22} > \sigma_{33}$) with no figure rotation ($\overline{v_{\phi_3}} = 0$), left picture, is turned into a configuration flattened by figure rotation [$\overline{v_{\phi_3}} = (\sigma_{11}^2 - \sigma_{33}^2)^{1/2} = (\sigma_{22}^2 - \sigma_{33}^2)^{1/2}$] with isotropic velocity component distribution ($\sigma'_{11} = \sigma'_{22} = \sigma_{33}$), right picture. The symbol, \mathcal{U} , denotes figure rotation around the related principal axis.

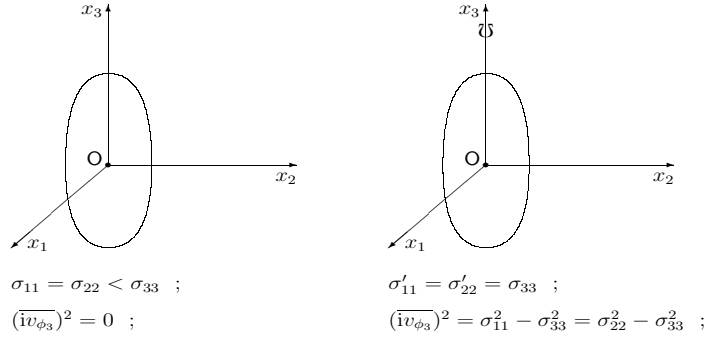


Figure 2: After imparting a convenient amount of real and imaginary figure rotation, $\overline{v_{\phi_3}}$ and $\overline{iv_{\phi_3}}$, with respect to the x_3 principal axis, to the system, and reversing real tangential velocity components on one half particles, a configuration elongated by anisotropic velocity component distribution ($\sigma_{11} = \sigma_{22} < \sigma_{33}$) with no figure rotation ($\overline{v_{\phi_3}} = 0$), left picture, is turned into a configuration elongated by (imaginary) figure rotation [$\overline{iv_{\phi_3}} = (\sigma_{11}^2 - \sigma_{33}^2)^{1/2} = (\sigma_{22}^2 - \sigma_{33}^2)^{1/2}$] with isotropic velocity component distribution ($\sigma'_{11} = \sigma'_{22} = \sigma_{33}$), right picture. The symbol, \mathcal{U} , denotes figure rotation around the related principal axis.

In the case under discussion ($r = 3$), an isotropic velocity component distribution after the reversion process implies the validity of Eqs. (105), (106), and (107), where the tangential velocity components are imaginary ($\sigma_{\ell\ell} < \sigma_{33}$), and the configuration after the reversion process is completely defined.

4.5 Tangential velocity component reversion in the general case

In the general case of anisotropic velocity component distribution ($\sigma_{11} \neq \sigma_{22} \neq \sigma_{33}$) and figure rotation around each principal axis ($\widetilde{\Omega}_1 \neq \widetilde{\Omega}_2 \neq \widetilde{\Omega}_3 \neq 0$), with regard to the $(O x_s x_t)$ principal plane, let the tangential velocity component reversion process be applied to one half particles in such a way no figure rotation around the x_r principal axis occurs, $\widetilde{\Omega}_r = 0$. Keeping in mind Eqs. (50)-(53), the related energy changes read:

$$(T_{\text{rdm}})_{\phi_r \phi_r} \rightarrow (T_{\text{rdm}})_{\phi_r \phi_r} + \frac{1}{2} M (\overline{v_{\phi_r}})^2 ; \quad (108)$$

$$(T_{\text{sys}})_{\phi_r \phi_r} \rightarrow (T_{\text{sys}})_{\phi_r \phi_r} - \frac{1}{2} M (\overline{v_{\phi_r}})^2 = 0 ; \quad (109)$$

$$\overline{v_{\phi_r}} = (\overline{v_{\phi_r}})_{N/2} = \frac{1}{N} \sum_{i=1}^N v_{\phi_r}^{(i)} ; \quad (110)$$

$$\Delta(T_{\text{sys}})_{\phi_r \phi_r} = -\Delta(T_{\text{rdm}})_{\phi_r \phi_r} = -\frac{1}{2} M (\overline{v_{\phi_r}})^2 = -\frac{1}{2} M R_{Gr}^2 (\widetilde{\Omega}_r)^2 ; \quad (111)$$

$$[(T_{\text{rdm}})_{\phi_r \phi_r}]_{\ell\ell} + [(T_{\text{rdm}})_{w_r w_r}]_{\ell\ell} = \frac{1}{2} M \sigma_{\ell\ell}^2 ; \quad \ell = 1, 2 ; \quad (112)$$

$$(T_{\text{rdm}})_{rr} = \frac{1}{2} M \sigma_{rr}^2 ; \quad (113)$$

where the rms velocity components, $\sigma_{\ell\ell}^2$ and σ_{rr}^2 , are related to the initial configuration.

The changes in anisotropy parameters, $\zeta_{pp} = \sigma_{pp}^2 / \sigma^2$, read:

$$\zeta_{\ell\ell} \rightarrow \frac{\zeta_{\ell\ell} + (1/2) \Delta \zeta_{\ell\ell}}{1 + \Delta \zeta_{\ell\ell}} ; \quad \ell = s, t ; \quad (114a)$$

$$\zeta_{rr} \rightarrow \frac{\zeta_{rr}}{1 + \Delta \zeta_{\ell\ell}} ; \quad (114b)$$

$$\Delta\zeta_{\ell\ell} = \frac{(\overline{v_{\phi_r}})^2}{\sigma^2} \quad ; \quad \ell = s, t \quad ; \quad (114c)$$

where the rms velocity, σ^2 , is related to the initial configuration.

The application of the above procedure to the x_1 and x_2 principal axes, makes the transition from an initial configuration with rms velocity components, $\sigma_{11}^2, \sigma_{22}^2, \sigma_{33}^2$, and figure rotation around the principal axes, $\widetilde{\Omega}_1, \widetilde{\Omega}_2, \widetilde{\Omega}_3$, to a final configuration with rms velocity components, $(\sigma'_{11})^2, (\sigma'_{22})^2, (\sigma'_{33})^2$, and figure rotation around the principal axes, 0, 0, $\widetilde{\Omega}_3$. The related energy changes read:

$$(T_{\text{rdm}})_{11} \rightarrow (T_{\text{rdm}})_{11} + \frac{1}{4}M(\overline{v_{\phi_2}})^2 \quad ; \quad (115)$$

$$(T_{\text{rdm}})_{22} \rightarrow (T_{\text{rdm}})_{22} + \frac{1}{4}M(\overline{v_{\phi_1}})^2 \quad ; \quad (116)$$

$$(T_{\text{rdm}})_{33} \rightarrow (T_{\text{rdm}})_{33} + \frac{1}{4}M[(\overline{v_{\phi_1}})^2 + (\overline{v_{\phi_2}})^2] \quad ; \quad (117)$$

$$T_{\text{rdm}} \rightarrow T_{\text{rdm}} + \frac{1}{2}M[(\overline{v_{\phi_1}})^2 + (\overline{v_{\phi_2}})^2] \quad ; \quad (118)$$

$$(T_{\text{rdm}})_{pp} = \frac{1}{2}M\sigma_{pp}^2 \quad ; \quad p = 1, 2, 3 \quad ; \quad (119)$$

and the related changes in anisotropy parameters are:

$$\zeta_{11} \rightarrow \frac{\zeta_{11} + (1/2)\Delta\zeta_{11}}{1 + \Delta\zeta_{11} + \Delta\zeta_{22}} \quad ; \quad (120a)$$

$$\zeta_{22} \rightarrow \frac{\zeta_{22} + (1/2)\Delta\zeta_{22}}{1 + \Delta\zeta_{11} + \Delta\zeta_{22}} \quad ; \quad (120b)$$

$$\zeta_{33} \rightarrow \frac{\zeta_{33} + (1/2)[\Delta\zeta_{11} + \Delta\zeta_{22}]}{1 + \Delta\zeta_{11} + \Delta\zeta_{22}} \quad ; \quad (120c)$$

$$\Delta\zeta_{11} = \frac{(\overline{v_{\phi_2}})^2}{\sigma^2} \quad ; \quad \Delta\zeta_{22} = \frac{(\overline{v_{\phi_1}})^2}{\sigma^2} \quad ; \quad ; \quad (120d)$$

the above results may be restricted to a single statement.

Theorem 3. *Given a R fluid, a convenient application of the tangential velocity component reversion process makes an adjoint configuration where figure rotation occurs around a single principal axis, that is a R3 fluid.*

Accordingly, the results valid for R3 fluids (C07) may be extended to the general case of R fluids.

5 Discussion

As suggested in earlier attempts (Caimmi 1996; C06; C07), the equivalence between a variation in figure rotation and in anisotropy excess, may provide a useful tool for the description of collisionless fluids. The discussion here shall be focused on the spin parameter (Peebles 1969, 1971):

$$\lambda^2 = -\frac{J^2 E}{G^2 M^5} ; \quad (121)$$

where G is the gravitation constant, M the total mass, J the total angular momentum, and E the total energy. The above formulation includes four possibilities, namely (i) real rotation ($J^2 \geq 0$) and bound system ($E < 0$), which is the sole currently used in literature; (ii) imaginary rotation ($J^2 < 0$) and bound system ($E < 0$); (iii) real rotation ($J^2 \geq 0$) and unbound system ($E \geq 0$); (iv) imaginary rotation ($J^2 < 0$) and unbound system ($E \geq 0$). Accordingly, the spin parameter attains real values in cases (i) and (iv), and imaginary values in cases (ii) and (iii).

In the light of the current model, oblate-like and prolate-like configurations belong to cases (i) and (ii) outlined above, while cases (iii) and (iv) represent unbound structures for which the virial equations do not hold. Then the comparison with observations and/or computations, must be restricted to bound configurations.

The spin parameter distribution is usually fitted using a lognormal distribution (e.g., van den Bosh 1998; Gardner 2001; Ballin & Steinmetz 2005; Hernandez et al. 2007) or, in general, dependent on $\log \lambda$ (e.g., Bett et al. 2007), in dealing with real rotation. The inclusion of imaginary rotation would imply use of λ^2 instead of λ as independent variable, allowing for both positive (real rotation) and negative (imaginary rotation) values.

The following procedure should be followed for calculating λ^2 from observations and/or computations: (1) determine the inertia tensor and the principal axes of inertia for an assigned matter distribution; (2) determine the potential-energy tensor; (3) determine the anisotropy parameters using the generalized virial equations; (4) perform the reversion process with respect to two principal axes of inertia to leave figure rotation around the third one (R3 fluid); (5) convert the anisotropy excess into (real or imaginary) figure rotation to obtain isotropic velocity component distribution ($\zeta_{11} = \zeta_{22} = \zeta_{33}$); (6) evaluate the spin parameter; (7) act as already done for all the sample

objects; (8) determine the distribution of the spin parameter, $P(\lambda^2)$, with respect to the sample of adjoints configurations, where the velocity component distribution is isotropic.

The related results could provide further insight on the formation and the evolution of large-scale collisionless fluids, such as galaxies and galaxy clusters.

6 Conclusion

A theory of collisionless fluids has been developed in a unified picture, where nonrotating ($\widetilde{\Omega}_1 = \widetilde{\Omega}_2 = \widetilde{\Omega}_3 = 0$) figures with isotropic ($\sigma_{11} = \sigma_{22} = \sigma_{33}$) random velocity component distributions and rotating ($\widetilde{\Omega}_1 \neq \widetilde{\Omega}_2 \neq \widetilde{\Omega}_3$) figures with anisotropic ($\sigma_{11} \neq \sigma_{22} \neq \sigma_{33}$) random velocity component distributions, make adjoints configurations to the same system. R fluids have been defined as ideal, self-gravitating fluids satisfying the virial theorem assumptions (e.g., LL66, Chap. II, § 10; C07), in presence of figure rotation around each principal axis of inertia.

To this aim, mean and rms angular velocities and mean and rms tangential velocity components have been expressed, by weighting on the moment of inertia and the mass, respectively. The figure rotation has been defined as the mean angular velocity, weighted on the moment of inertia, with respect to a selected axis.

The generalized tensor virial equations (Caimmi & Marmo 2005) have been formulated for R fluids and further attention has been devoted to axisymmetric configurations where, for selected coordinate axes, a variation in figure rotation has to be counterbalanced by a variation in anisotropy excess and vice versa.

A microscopical analysis of systematic and random motions has been performed under a few general hypotheses, by reversing the sign of tangential or axial velocity components of an assigned fraction of particles, leaving the distribution function and other parameters unchanged (Meza 2002).

The application of the reversion process to tangential velocity components, has been found to imply the conversion of random motion rotation kinetic energy into systematic motion rotation kinetic energy. The application of the reversion process to axial velocity components, has been found to imply the conversion of random motion translation kinetic energy into sys-

tematic motion translation kinetic energy, and the loss related to a change of reference frame has been expressed in terms of systematic motion (imaginary) rotation kinetic energy.

A number of special situations have been investigated with further detail. It has been found that a R fluid always admits an adjoint configuration where figure rotation occurs around only one principal axis of inertia (R3 fluid), which implies that all the results related to R3 fluids (Caimmi 2007) may be extended to R fluids.

Finally, a procedure has been sketched for deriving the spin parameter distribution (including imaginary rotation) from a sample of observed or simulated large-scale collisionless fluids i.e. galaxies and galaxy clusters.

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